

ACTIONS OF FINITE GROUPS AND SMOOTH FUNCTIONS ON
SURFACES

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ABSTRACT. Let $f : M \rightarrow \mathbb{R}$ be a Morse function on a smooth closed surface, V be a connected component of some critical level of f , and \mathcal{E}_V be its atom. Let also $\mathcal{S}(f)$ be a stabilizer of the function f under the right action of the group of diffeomorphisms $\text{Diff}(M)$ on the space of smooth functions on M , and $S_V(f) = \{h \in \mathcal{S}(f) \mid h(V) = V\}$. The group $S_V(f)$ acts on the set $\pi_0 \partial \mathcal{E}_V$ of connected components of the boundary of \mathcal{E}_V . Therefore we have a homomorphism $\phi : \mathcal{S}(f) \rightarrow \text{Aut}(\pi_0 \partial \mathcal{E}_V)$. Let also $G = \phi(\mathcal{S}(f))$ be the image of $\mathcal{S}(f)$ in $\text{Aut}(\pi_0 \partial \mathcal{E}_V)$.

Suppose that the inclusion $\partial \mathcal{E}_V \subset M \setminus V$ induces a bijection $\pi_0 \partial \mathcal{E}_V \rightarrow \pi_0(M \setminus V)$. Let H be a subgroup of G . We present a sufficient condition for existence of a section $s : H \rightarrow S_V(f)$ of the homomorphism ϕ , so, the action of H on $\partial \mathcal{E}_V$ lifts to the H -action on M by f -preserving diffeomorphisms of M .

This result holds for a larger class of smooth functions $f : M \rightarrow \mathbb{R}$ having the following property: for each critical point z of f the germ of f at z is smoothly equivalent to a homogeneous polynomial $\mathbb{R}^2 \rightarrow \mathbb{R}$ without multiple linear factors.

1. INTRODUCTION

Let M be a smooth compact surface. The group of diffeomorphisms $\mathcal{D}(M)$ of M acts on the space of smooth functions $C^\infty(M)$ by the rule

$$(1.1) \quad C^\infty(M) \times \mathcal{D}(M) \rightarrow C^\infty(M), \quad (f, h) = f \circ h.$$

The set $\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\}$ is called the *stabilizer* of the function f under action (1.1). Endow $C^\infty(M)$ and $\mathcal{D}(M)$ with the corresponding Whitney topologies. The topology on $\mathcal{D}(M)$ induces a certain topology on the stabilizer $\mathcal{S}(f)$.

Let $\mathcal{F}(M) \subset C^\infty(M, \mathbb{R})$ be the set of smooth functions satisfying the following two conditions:

- (B) the function f takes a constant value at each connected component of ∂M , and all critical points of f belong to the interior of M ;
- (P) for each critical point x of f the germ (f, x) of f at x is smoothly equivalent to some homogeneous polynomial $f_x : \mathbb{R}^2 \rightarrow \mathbb{R}$ without multiple linear factors.

It is well-known that each homogeneous polynomial $f_x : \mathbb{R}^2 \rightarrow \mathbb{R}$ splits into a product of linear and irreducible over \mathbb{R} quadratic factors. Condition (P) means that

$$(1.2) \quad f_x = \prod_{i=1}^n L_i \cdot \prod_{j=1}^m Q_j^{q_j},$$

where $L_i(x, y) = a_i x + b_i y$ is a linear form, $Q_j = c_j x^2 + d_j xy + e_j y^2$ is an irreducible quadratic form such that $L_i/L_{i'} \neq \text{const}$ for $i \neq i'$, and $Q_j/Q_{j'} \neq \text{const}$ for $j \neq j'$. So, if $\deg f_x \geq 2$, then 0 is an isolated critical point of f_x .

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Recall that if $f : (\mathbb{C}, 0) \rightarrow (\mathbb{R}, 0)$ is a germ of C^∞ -function such that $0 \in \mathbb{R}^2$ is an isolated critical point of f , then there is a germ of a homeomorphism $h : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ such that

$$f_x \circ h(z) = \begin{cases} \pm|z|^2, & \text{if } 0 \text{ is a local extremum, [4],} \\ \operatorname{Re}(z^n), & \text{for some } n \in \mathbb{N} \text{ otherwise, [13].} \end{cases}$$

If 0 is not a local extreme, then the number n does not depend of a particular choice of h . In this case the point 0 will be called a *generalized n -saddle*, or simply an *n -saddle*. The number n corresponds to the number of linear factors in (1.2). Examples of level sets foliations near isolated critical points are given in Fig. 1.1.

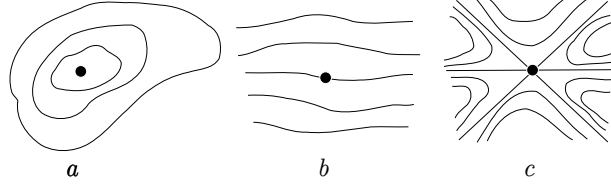


FIGURE 1.1. Level set foliations in neighborhoods of isolated points ((a) local extreme, (b) 1-saddle, (c) 3-saddle)

Let $\operatorname{Morse}_\partial(M)$ be the space of Morse functions on M , which satisfy condition (B), $f \in \operatorname{Morse}_\partial(M)$, and x be a critical point of f . Then, by Morse Lemma, there exists a coordinate system (t, s) near x such that the function f has one of the following forms $f(s, t) = \pm s^2 \pm t^2$, which are, obviously, homogeneous polynomials without multiple factors. This implies that $\operatorname{Morse}_\partial(M)$ is a subspace of $\mathcal{F}(M)$.

Let $f \in \mathcal{F}(M)$ be a smooth function and $c \in \mathbb{R}$ be a real number. A connected component C of the level set $f^{-1}(c)$ is called *critical* if it contains at least one critical point, otherwise, C is called *regular*. Let Δ be a foliation of M into connected components of level sets of f . It is well-known that the quotient-space M/Δ has a structure of 1-dimensional CW complex. The space M/Δ is called the Kronrod-Reeb graph, or simply, KR-graph of f . We will denote it by Γ_f . Let $p_f : M \rightarrow \Gamma_f$ be a projection of M onto Γ_f . Then vertices of Γ_f correspond to connected components of critical level sets of the function f .

It should be noted that the function $f \in \mathcal{F}(M)$ can be represented as the composition

$$f = \phi \circ p_f : M \xrightarrow{p_f} \Gamma_f \xrightarrow{\phi'} \mathbb{R},$$

where ϕ' is the map induced by f . Let $h \in \mathcal{S}(f)$. Then $f \circ h = f$, and we have $h(f^{-1}(t)) = f^{-1}(t)$ for all $t \in \mathbb{R}$. Hence h interchanges connected components of level sets of the function f and therefore it induces an automorphism $\rho(h)$ of KR-graph Γ_f such that the following diagram is commutative:

In other words, we have a homomorphism $\rho : \mathcal{S}(f) \rightarrow \operatorname{Aut}(\Gamma_f)$. Let $G = \rho(\mathcal{S}(f))$ be the image of $\mathcal{S}(f)$ in $\operatorname{Aut}(\Gamma_f)$. It is easy to show that the group G is finite.

Let v be a vertex of Γ_f and $G_v = \{g \in G \mid g(v) = v\}$ be the stabilizer of v under the action of G on Γ_f . An arbitrary connected closed G_v -invariant neighborhood of v in Γ_f containing no other vertices of Γ_f will be called a *star* of v . We denote it by $\operatorname{st}(v)$.

The set $G_v^{\operatorname{loc}} = \{g|_{\operatorname{st}(v)} \mid g \in G\}$ which consists of restrictions of elements of G_v onto the star $\operatorname{st}(v)$ is a subgroup of $\operatorname{Aut}(\operatorname{st}(v))$. This group will be called a *local stabilizer* of v . Let also $r : G_v \rightarrow G_v^{\operatorname{loc}}$ be the map defined by $r(g) = g|_{\operatorname{st}(v)}$ for $g \in G_v$, i.e., r is the restriction map.

Let v be a vertex of Γ_f , and $V = p_f^{-1}(v)$ be the corresponding connected component of the critical level set $f^{-1}(p_f(v))$.

Definition 1.1. A vertex v of the graph Γ_f will be called **special** if there is a bijection between connected components of $\text{st}(v) \setminus v$ and $M \setminus V$. The corresponding connected component $V = p_f^{-1}(v)$ will be called **special**.

It follows from definition of KR-graph Γ_f that for a special vertex v there is a 1–1 correspondence between connected components of complement to v in $\text{st}(v)$ and connected components of $\Gamma_f \setminus v$.

Note that a special component V gives a partition Ξ of the surface M whose 0-dimensional elements are vertices of V , 1-dimensional elements are edges of V , and 2-dimensional elements are connected components of complement of V in M . Since M is compact, it follows that Ξ has a finite number of elements in each dimension.

2. MAIN RESULT

Let $f \in \mathcal{F}(M)$. Suppose that its Kronrod-Reeb graph Γ_f contains a special vertex v , and V be the special component of level set of f which corresponds to v .

Let $\mathcal{S}_V(f) = \{h \in \mathcal{S}(f) \mid h(V) = V\}$ be a subgroup of $\mathcal{S}(f)$ leaving V invariant. It is easy to see that $\rho(\mathcal{S}_V(f)) \subset G_v$. We denote by ϕ the map

$$\phi = r \circ \rho : \mathcal{S}_V(f) \xrightarrow{\rho} G_v \xrightarrow{r} G_v^{\text{loc}}.$$

Let H be a subgroup of G_v^{loc} and $\mathcal{H} = \phi^{-1}(H)$ be a subgroup of $\mathcal{S}_V(f)$. We will say that the group \mathcal{H} has property (C) if the following conditions hold.

- (C) Let $h \in \mathcal{H}$, and E be a 2-dimensional element of Ξ . Suppose that $h(E) = E$. Then $h(e) = e$ for all other $e \in \Xi$, and the map h preserves orientation of each element of Ξ .

Lemma 2.1. If $\mathcal{H} = \phi^{-1}(H)$ has property (C), then H acts on the set of all elements of the partition Ξ . Moreover this action is free on the set of 2-dimensional elements of Ξ .

Proof. Let $g \in H$, and $h \in \mathcal{H}$ be a diffeomorphism such that $\phi(h) = g$. Define the map $\tau : H \times \Xi \rightarrow \Xi$ by the following rule

$$\tau(g, e) = h(e), \quad e \in \Xi.$$

We claim that this definition does not depend of a particular choice of such h . Let $h_1, h_2 \in \mathcal{H}$ be diffeomorphisms such that $\phi(h_1) = \phi(h_2)$. Then $\phi(h_1 \circ h_2^{-1}) = 1_H$, where 1_H be the unit of H . By definition of the unit 1_H , we have $(h_1 \circ h_2^{-1})(E) = E$ for each 2-dimensional component E of Ξ . Then, by condition (C), $(h_1 \circ h_2^{-1})(e) = e$ for other $e \in \Xi$. Hence $h_1(e) = h_2(e)$. So, the map τ is well-defined. It is easy to see that $\tau(1_H, e) = e$, where 1_H is the unit of H , and $\tau(g_1, \tau(g_2, e)) = \tau(g_1 \circ g_2, e)$ for each $g_1, g_2 \in H$, and $e \in \Xi$. Thus τ is an H -action on Ξ .

Suppose $h \in \mathcal{H}$ is such that $h(E) = E$ for some 2-dimensional component E of Ξ . Then, by condition (C), $h(E') = E'$ for each 2-dimensional component E' of Ξ . Hence, $h = \text{id}$, so the H -action on the set of 2-dimensional components of Ξ is free. \square

Thus condition (C) implies that $H \ll \text{combinatorially} \gg$ acts of M i.e., it ensures invariance of the partition Ξ under the action of H on M . Our aim is to prove that in fact this $\ll \text{combinatorial} \gg$ action is induced by a real action of H on M by diffeomorphisms preserving f .

Namely the following theorem holds.

Theorem 2.2. Suppose $f \in \mathcal{F}(M)$ is such that its KR-graph Γ_f contains a special vertex v , and G_v^{loc} be the local stabilizer of v . Let also H be a subgroup of G_v^{loc} , and $\mathcal{H} = \phi^{-1}(H)$ be a subgroup of $\mathcal{S}_V(f)$ satisfying condition (C). Then there exists a section $s : H \rightarrow \mathcal{H}$ of the map ϕ , i.e., the map s is a homomorphism satisfying the condition $\phi \circ s = \text{id}_H$.

Group actions which have the property of invariance of some partition of the surfaces are studied by Bolsinov and Fomenko [1], Brailov [2], Brailov and Kudryavtseva [3], Kudryavtseva [5], Maksymenko [11], Kudryavtseva and Fomenko [6, 7].

2.3. Structure of the paper. In Section 3 we recall the definitions and statements that will be used in the text. The topological structure of the atom $\mathcal{E}_{V,\varepsilon}$ which corresponds to V is described in Section 4. In section 5, we construct an H -action on the surface M .

3. SYMMETRIES OF HOMOGENEOUS POLYNOMIALS

Let $f_z : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a homogeneous polynomial without multiple linear factors. Suppose the origin $0 \in \mathbb{R}^2$ is not a local extreme for f_z . Let also $\mathcal{L}(f_z)$ be a group of orientation preserving linear automorphisms $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $f_z \circ h = f_z$. The following lemma holds:

Lemma 3.1. ([10], Section 6). *After some linear change of coordinates one can assume that*

- (1) *if $\deg f_z = 2$, then the group $\mathcal{L}(f_z)$ consists of the linear transformations of the following form*

$$\pm \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, \quad a > 0,$$

see [10, Section 6, case (B)];

- (2) *if $\deg \geq 3$, then the group of $\mathcal{L}(f_z)$ is a finite cyclic subgroup of $\mathrm{SO}(2)$, [10, Section 6, case (E)].*

We will also need the following lemma:

Lemma 3.2. ([10], Corollary 7.4). *Let $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be a germ of a diffeomorphism $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ at $0 \in \mathbb{R}^2$, and $T_0 h$ be its tangent map at $0 \in \mathbb{R}^2$. If $f_z \circ h = f_z$, then $f_z \circ T_0 h = f_z$.*

Proof. For the sake of completeness we will recall a short proof from [10].

Assume that the polynomial f_z is a homogeneous function of degree k , i.e., $f_z(tx) = t^k f_z(x)$ for $t \geq 0$ and $x \in \mathbb{R}^2$. Then

$$f_z(x) = \frac{f_z(tx)}{t^k} = \frac{f_z(h(tx))}{t^k} = f_z\left(\frac{h(tx)}{t}\right) \xrightarrow{t \rightarrow 0} (f_z \circ T_0 h)(x).$$

Lemma 3.2 is proved. \square

4. TOPOLOGICAL STRUCTURE OF THE ATOM $\mathcal{E}_{V,a}$

Let f be a smooth function from $\mathcal{F}(M)$, $\varepsilon_1 > 0$, $c \in \mathbb{R}$, and V be a connected component of some critical level $f^{-1}(c)$ of f .

Let also \mathcal{E} be a connected component of $f^{-1}([c - \varepsilon_1, c + \varepsilon_1])$, which contains V . Assume that the boundary $\partial \mathcal{E}$ consists of $n + k$ connected components A_i , $i = 1, 2, \dots, n + k$, i.e., $\partial \mathcal{E} = \bigcup_{i=1}^n A_i \cup \bigcup_{j=1}^k A_{-j}$. Since $f \in \mathcal{F}(M)$, it follows that $f|_{\mathcal{E}}$ belongs to $\mathcal{F}(\mathcal{E})$, and so, by (B), $f|_{\mathcal{E}}$ takes a constant value at each connected component of the boundary $\partial \mathcal{E}$. Assume that $f(A_i) = c_i \in [c, c + \varepsilon_1]$, $i \geq 1$, and $f(A_i) = d_i \in [c - \varepsilon_1, c]$, $i \leq 1$. Put $c' = \min\{c_i\}$ and $d' = \max\{d_i\}$. Fix $a > 0$ such that $[c - a, c + a] \subset [d', c']$.

A connected component of $f^{-1}([c - a, c + a])$, which contains V will be called an *atom* of V and denoted by $\mathcal{E}_{V,a}$.

Let H be a subgroup of G_v^{loc} and $\mathcal{H} = \phi^{-1}(H) \subset S_V(f)$. We will need the following lemma.

Lemma 4.1. *Let $\mathcal{E}_{V,a}$ be an atom of a special critical component V , A be a connected component of $\partial\mathcal{E}_{V,a}$, and $h \in \mathcal{H}$. Assume that the group \mathcal{H} has property (C). If $h|_{\mathcal{E}_{V,a}}(A) = A$, then h preserves the orientation of A .*

Proof. Fix a Riemannian metric $\langle \cdot, \cdot \rangle$ on M . Let ∇f be a gradient vector field of the function f in this Riemannian metric. Let also Q be a set of points $x \in A$ such that there exists an integral curve c_x of ∇f , which joins the point x with some point $y_x \in V$. Then Q is a union of open intervals in A , and the map $\psi : Q \rightarrow V$, $\psi(x) = y_x$ is an embedding. The image of $\psi(Q)$ is a cycle in V . So, the connected component A of $\partial\mathcal{E}_{V,a}$ defines the cycle γ_A in V . Moreover the orientation of A induces the orientation of γ_A and vice versa, see [12].

Assume that \mathcal{H} has property (C). Let $h \in \mathcal{H}$ and E be a 2-dimensional element of Ξ such that $h(E) = E$. Then by (C), $h(e) = e$ for all other $e \in \Xi$. In particular $h(\gamma_A) = \gamma_A$ and h preserves orientation of γ_A . Then $h(A) = A$, and h preserves orientation of A . \square

5. PROOF OF THEOREM 2.2

Suppose $f \in \mathcal{F}(M)$ is such that its KR-graph contains a special vertex v , $V = p_f^{-1}(v)$ be the corresponding special component of some level set, which corresponds to v , and G_v^{loc} be the local stabilizer of v .

Let H be a subgroup of G_v^{loc} such that $\mathcal{H} = \phi^{-1}(H)$ has property (C). We will construct a lifting of the H -action on $\text{st}(v)$ to the action $\Sigma : H \times M \rightarrow M$ of the group H on the surface M .

By Lemma 2.1 there is an action $\sigma^0 : H \times V \rightarrow V$ of H on the set of vertices of V defined by the rule:

$$\sigma^0(g, z) = h(z),$$

where $h \in \mathcal{H}$ is any diffeomorphism such that $\phi(h) = g$.

Step 1. Now we will extend the action σ^0 to the H -action σ^1 on the set of neighborhoods of vertices of V . Assume that the action σ^0 has s orbits $V_r = \{z_{r0}, z_{r1}, \dots, z_{rk(r)}\}$ for some $k(r) \in \mathbb{N}$, $r = 1, 2, \dots, s$, and let $V = \bigcup_{r=1}^s V_r$ be the union of vertices of V .

Then, by definition of the class $\mathcal{F}(M)$, for each $r = 1, 2, \dots, s$ there exists a chart (U_{r0}, q_{r0}) which contains z_{r0} such that the map $f \circ q_{r0}^{-1} = f_r$ is a homogeneous polynomial without multiple linear factors. We can also assume that $q_{r0}(U_{r0}) \subset \mathbb{R}^2$ is a 2-disk with the center at $0 \in \mathbb{R}^2$ and radius ε , and the group $\mathcal{L}(f_r)$ has the properties described in Lemma 3.1. Fix any diffeomorphisms

$$(5.1) \quad h_{ri} \in \mathcal{H} \quad \text{such that} \quad h_{ri}(z_{r0}) = z_{ri}, \quad i = 1, 2, \dots, k(r),$$

and define charts (U_{ri}, q_{ri}) for the points z_{ri} , $i = 1, 2, \dots, k(r)$ in the following way:

- $U_{ri} = h_{ri}(U_{r0})$;
- the map q_{ri} is defined from the diagram:

$$\begin{array}{ccc} U_{r0} & \xrightarrow{h_{ri}} & U_{ri} \\ q_{r0} \downarrow & \swarrow q_{ri} & \\ q_{r0}(U_{r0}) & & \end{array}$$

$$\text{i.e., } q_{ri} = q_{r0} \circ h_{ri}^{-1}.$$

Reducing ε , we can assume that $U_{ri} \cap U_{rj} = \emptyset$ for $i \neq j$.

Thus the chart (U_{ri}, q_{ri}) is chosen so that the map $f \circ q_{ri}^{-1} : q_{ri}(U_{ri}) \rightarrow \mathbb{R}$ is a homogeneous polynomial without multiple linear factors which coincides with given polynomial f_r for the chart (U_{r0}, q_{r0}) . We also put $U_r = \bigcup_{i=0}^{k(r)} U_{ri}$, and $U = \bigcup_{r=1}^s U_r$.

Lemma 5.1. *There exist a homomorphism $\lambda_1 : \mathcal{H} \rightarrow \text{Diff}(\mathcal{U})$ and a monomorphism $\chi_1 : H \rightarrow \text{Diff}(\mathcal{U})$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\lambda_1} & \text{Diff}(\mathcal{U}) \\ \phi \downarrow & \nearrow \chi_1 & \\ H & & \end{array}$$

Proof. (1) First we construct a map λ_1 . Let $h \in \mathcal{H}$ be such that $h(z_{ri}) = z_{rj}$ for some $i, j = 0, 1, \dots, k(r)$ and $r = 1, 2, \dots, s$. Let also $\gamma_h = q_{rj} \circ h \circ q_{ri}^{-1}$ be a diffeomorphism of $q_{r0}(U_{r0})$. It is easy to see that the map γ_h preserves the polynomial f_r . By Lemma 3.2 the tangent map $T_0\gamma_h$ also preserves the polynomial f_r , so $T_0\gamma_h \in \mathcal{L}(f_r)$. Define a linear map $A \in \mathcal{L}(f_r)$ as follows: if $\deg f_r = 2$, then, by Lemma 3.1,

$$T_0\gamma_h = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, \quad a \neq 0,$$

and we set

$$A_h = \text{sign}(a) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $\deg f_z \geq 3$, then by assumption and Lemma 3.1, $\mathcal{L}(f_z)$ is a cyclic subgroup of $\text{SO}(2)$. In this case we put

$$A_h = T_0\gamma_h.$$

We define the diffeomorphism $\lambda_1(h) \in \text{Diff}(\mathcal{U})$ by the rule:

$$(5.2) \quad \lambda_1(h)|_{U_{ri}} = q_{rj}^{-1} \circ A_h \circ q_{ri}.$$

(2) Now we prove that the map λ_1 is a homomorphism. Suppose $h_1, h_2 \in \mathcal{H}$ are such that $h(z_{ri}) = z_{rj}$ and $h(z_{rj}) = z_{rk}$. By (5.2), we have

$$\lambda_1(h_1)|_{U_{ri}} = q_{rj}^{-1} \circ A_{h_1} \circ q_{ri}, \quad \lambda_1(h_2)|_{U_{rj}} = q_{rk}^{-1} \circ A_{h_2} \circ q_{rj},$$

and

$$\lambda_1(h_2)|_{U_{rj}} \circ \lambda_1(h_1)|_{U_{ri}} = q_{rk}^{-1} \circ A_{h_2} \circ A_{h_1} \circ q_{ri}.$$

On the other hand, we have

$$\lambda_1(h_2 \circ h_1) = q_{rk}^{-1} \circ A_{h_2 \circ h_1} \circ q_{ri}.$$

It follows from the definition of the linear map A_h , that $A_{h_2 \circ h_1} = A_{h_2} \circ A_{h_1}$. Hence

$$\lambda_1(h_2 \circ h_1) = \lambda_1(h_2) \circ \lambda_1(h_1).$$

So, the map λ_1 is a homomorphism.

(3) Let $g \in H$ and $h \in \mathcal{H}$ be such that $\phi(h) = g$. Then we define the map $\chi_1 : H \rightarrow \text{Diff}(\mathcal{U})$ by the rule

$$\chi_1(g) = \lambda_1(h).$$

Obviously that χ_1 is a homomorphism. It remains to prove that the map χ_1 is a monomorphism. It is sufficient to check that $\text{Ker}\chi_1 = \text{Ker}\psi$, i.e., $\lambda_1(h) = \text{id}_{\mathcal{U}}$ iff h trivially acts on the set of 2-dimensional elements of Ξ .

Suppose that h trivially acts on the set of 2-dimensional elements of Ξ . By condition (C), h trivially acts on set of vertices and edges of V . Since $h(z_{ri}) = z_{ri}$ for all $i = 0, 1, \dots, k(r)$ and $r = 1, 2, \dots, s$, it follows from (5.2) that $\lambda_1(h) = \text{id}_{\mathcal{U}}$.

Suppose $h \in \mathcal{H}$ is such that $\lambda_1(h) = \text{id}_{\mathcal{U}}$. Then $h(e) = e$ for each edge e of V , and h preserves the orientation of e . Hence by Lemma 4.1, h leaves invariant each connected component of $\partial\mathcal{E}_{V,a}$ with its orientation. Therefore h trivially acts on the set of 2-dimensional elements of Ξ . \square

Let $\sigma^1 : H \times U \rightarrow U$ be a map defined by the formula

$$\sigma^1(g, x) = \chi_1(g)(x), \quad x \in U.$$

Since χ_1 is a homomorphism, it follows that σ^1 is an H -action on U .

Step 2. In this step we extend the action σ^1 to the H -action σ on the atom $\mathcal{E}_{V,a}$. We start with some preliminaries. Let (U_{ri}, q_{ri}) be the chart on M , which contains z_{ri} , defined above. The projection map q_{ri} induces the map $Tq_{ri} : TU_{ri} \rightarrow Tq_{ri}(U_{ri})$ between tangent bundles of U_{ri} and $q_{ri}(U_{ri}) \subset \mathbb{R}^2$. Fix a Riemannian metric $\langle \cdot, \cdot \rangle$ on M such that the following diagram is commutative

$$\begin{array}{ccc} TU_{ri} & \xrightarrow{Tq_{ri}} & Tq_{ri}(U_{ri}) \\ \nabla f|_{U_{ri}} \uparrow & & \uparrow \nabla f_r|_{q_{ri}(U_{ri})} \\ U_{ri} & \xrightarrow{q_{ri}} & q_{ri}(U_{ri}) \end{array}$$

where ∇f and ∇f_r are gradient fields of f and f_r in Riemannian metrics on M and on \mathbb{R}^2 respectively. Let also \mathbf{G} be the flow of ∇f on M .

Another description of the diffeomorphism $\lambda_1(h)$. Let $x \in U_{ri}$ be a point, $i = 0, 1, 2, \dots, k(r)$, $r = 1, 2, \dots, s$, and $y = \lambda_1(h)(x)$ be its image under $\lambda_1(h)$. Let also ω_x and ω_y be the trajectories of the gradient flow \mathbf{G} such that $x \in \omega_x$ and $y \in \omega_y$. Since $\lambda_1(h)$ preserves trajectories of the flow \mathbf{G} in U , it follows that $\lambda_1(h)(\omega_x \cap U_{ri}) = \omega_y \cap \lambda_1(h)(U_{ri})$. By definition of $\lambda_1(h)$ we have that $f(x) = f(y)$. In particular, if the trajectory ω_x intersects some edge R of V at some point x' , and $y' = \lambda_1(h)(x')$, then $y = f^{-1}(f(x)) \cap \omega_{y'}$, where $\omega_{y'}$ is the trajectory of \mathbf{G} , which passes through the point y' . Namely the image of x is uniquely defined by the image of the point x' .

By Lemma 2.1, the group H acts on the set of all edges R of V . Assume that this action has u orbits $R_r = \{R_{r0}, R_{r1}, \dots, R_{rn(u)}\}$ for some $n(u) \in \mathbb{N}$ and $r = 1, 2, \dots, u$. We also put $R = \bigcup_{r=1}^u R_r$. For each edge R_{ri} fix

- (I) a C^∞ -diffeomorphism $\ell_{ri} : (-1, 1) \rightarrow R_{ri}$ such that restrictions $\ell_{ri}|_{(-1, -1+\varepsilon)}$ and $\ell_{ri}|_{(1-\varepsilon, 1)}$ are isometries,

where ε is the radius of the disk $q_{r0}(U_{r0})$ defined in Step 1.

Lemma 5.2. *There exist a homomorphism $\lambda_2 : \mathcal{H} \rightarrow \text{Diff}(\mathcal{E}_{V,a})$ and a monomorphism $\chi_2 : H \rightarrow \text{Diff}(\mathcal{E}_{V,a})$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\lambda_2} & \text{Diff}(\mathcal{E}_{V,a}) \\ \phi \downarrow & \nearrow \chi_2 & \\ H & & \end{array}$$

and $\lambda_2(h)|_U = \lambda_1(h)|_{U \cap \mathcal{E}_{V,a}}$.

Proof. Let $h \in \mathcal{H}$. We will extend the diffeomorphism $\lambda_1(h)$ to a diffeomorphism $\lambda_2(h)$ of the atom $\mathcal{E}_{V,a}$. Let $x \in \mathcal{E}_{V,a}$ be any point. If $x \in U_{ri}$ for some $i = 0, 1, \dots, k(r)$, $r = 1, 2, \dots, s$, then we put $\lambda_2(h)(x) := \lambda_1(h)(x)$.

Suppose that $x \notin U$. Let ω_x be a trajectory of the flow \mathbf{G} passing through the point x . Then we have one of the following two cases: the trajectory ω_x either

- (1) intersects some edge R of V at a point, say y , or
- (2) converges to some vertex z of V .

In the case (1) let $R' = h(R)$, $\ell : (-1, 1) \rightarrow R$ and $\ell' : (-1, 1) \rightarrow R'$ be maps, defined by (I) for R and R' respectively, and

$$h' = \ell' \circ \ell^{-1} : R \xrightarrow{\ell^{-1}} (-1, 1) \xrightarrow{\ell'} R'.$$

Let also $y' = h'(y) \in R'$, $\omega_{y'}$ be the trajectory of \mathbf{G} , which passes through y' , and x' be a unique point in $\omega_{y'}$ such that $f(x) = f(x')$. Then we put $\lambda_2(h)(x) = x'$.

Consider the case (2). Let U be the neighborhood of z , defined in Step 1, $z' = \lambda_1(h)(z)$ be the corresponding point in $U' = \lambda_1(h)(U)$, $\omega_{z'}$ be the trajectory of \mathbf{G} such that $\omega_{z'} \cap U' = \lambda_1(h)(\omega_x \cap U)$, and x' be a unique point in $\omega_{z'}$ such that $f(x) = f(x')$. In this case we define $\lambda_2(h)$ by the rule: $\lambda_2(h)(x) = x'$.

By definition $\lambda_2(h)|_U = \lambda_1(h)|_{U \cap \mathcal{E}_{V,a}}$. Let $\chi_2 : H \rightarrow \text{Diff}(\mathcal{E}_{V,a})$ be the map defined as follows: for $g \in H$ and $h \in \mathcal{H}$ such that $\phi(h) = g$, we put $\chi_2(g) = \lambda_2(h)$. It is easy to check that the map λ_2 is a homomorphism. Moreover $\lambda_2(h) = \text{id}_{\mathcal{E}_{V,a}}$ iff $\lambda_1(h) = \text{id}_U$. Therefore χ_2 is a monomorphism. \square

Define the map $\sigma : H \times \mathcal{E}_{V,a} \rightarrow \mathcal{E}_{V,a}$ by the rule

$$\sigma(g, x) = \chi_2(g)(x).$$

Since χ_2 is the homomorphism, it follows that the map σ is an H -action on the atom $\mathcal{E}_{V,a}$.

Step 3. In this step we extend the H -action σ on the atom $\mathcal{E}_{V,a}$ to the H -action on the surface M . We start with some preliminaries. Let \mathbf{E} be a set of 2-dimensional elements of Ξ . By Lemma 2.1 the group H acts on the set \mathbf{E} . Assume that this action has y orbits $\mathbf{E}_r = \{E_{r0}, E_{r1}, \dots, E_{rk(r)}\}$, $i = 0, 1, \dots, k(r)$, and $r = 1, 2, \dots, y$. We also put $\mathbf{E} = \bigcup_{r=1}^y \mathbf{E}_r$. Fix diffeomorphisms $h_{ri} \in \mathcal{H}$ such that $h_{ri}(E_{r0}) = E_{ri}$.

Let $Y_r = E_{r0} \cap f^{-1}([-a, -a/2] \cup [a/2, a]) \cap \mathcal{E}_{V,a}$. Since v is a special vertex, it follows that the set Y_r is a cylinder. We put $Y_{ri} = h_{ri}(Y_r)$, and $Y = \bigcup_{r=1}^y \bigcup_{i=0}^{k(r)} Y_{ri}$.

We choose $a_1 > a$ such that the set \mathcal{E}_{V,a_1} is also an atom of V . Let

$$Z_r = E_{r0} \cap f^{-1}([-a_1, a/2] \cup [a/2, a_1]) \cap \mathcal{E}_{V,a_1}.$$

By definition, we have that $Y_r \subset Z_r$, and Z_r does not contain critical points of f . We also put $Z_{ri} = h_{ri}(Y_r)$ and $Z = \bigcup_{r=1}^y \bigcup_{i=0}^{k(r)} Z_{ri}$.

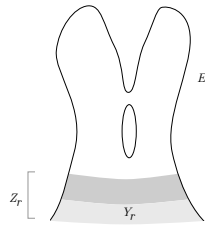


FIGURE 5.1. The 2-dimensional component E_{r0} , and its subsets Y_r and Z_r .

Fix a vector field F on Z such that its orbits coincide with connected components of level sets of the restriction $f|_Z$, and let \mathbf{F} be the flow of F . Then for each smooth function $\alpha \in C^\infty(M)$ we can define the following map

$$\mathbf{F}_\alpha : M \rightarrow M, \quad \mathbf{F}_\alpha(x) = \mathbf{F}(x, \alpha(x)).$$

Such maps have been studied in [8].

Since all orbits of \mathbf{F} are closed, it follows from [8, Theorem 19] that the map \mathbf{F}_α is a diffeomorphism, iff the Lie derivative $F\alpha$ of α along F satisfies the condition: $F\alpha > -1$. Moreover we have that $(\mathbf{F}_\alpha)^{-1} = \mathbf{F}_\xi$, where

$$(5.3) \quad \xi = -\alpha \circ \mathbf{F}_\alpha^{-1}.$$

Lemma 5.3. *For each $g \in H$ the map $\chi_2(g)$ extends to a diffeomorphism $\Sigma(g) \in \mathcal{S}(f)$, so that the correspondence $g \mapsto \Sigma(g)$ is a homomorphism $\Sigma : H \rightarrow \mathcal{S}(f)$.*

Proof. We will need the following two lemmas.

Lemma 5.4. *Let $g \in H$ and $h \in \mathcal{H}$ be such that $\phi(h) = g$, and $h(E_{r0}) = E_{ri}$. Then there exists a unique C^∞ -function $\xi_{ri} : Y_r \rightarrow \mathbb{R}$ such that*

$$\chi_2(g)|_{Y_r} = h_{ri}|_{Y_r} \circ \mathbf{F}_{\xi_{ri}} : Y_r \rightarrow Y_r.$$

In particular, the function ξ_{ri} depends only on g .

Lemma 5.5. *The diffeomorphism $\mathbf{F}_{\xi_{ri}}$ extends to a diffeomorphism $w_{ri} : E_{r0} \rightarrow E_{r0}$ such that $f \circ w_{ri} = f$ on E_{r0} .*

We prove Lemma 5.4 and Lemma 5.5 bellow, and now we will complete Theorem 2.2.

Define a diffeomorphism $\tilde{h}_{ri} : E_{r0} \rightarrow E_{ri}$ by the formula:

$$\tilde{h}_{ri} = h_{ri} \circ w_{ri}.$$

Let $h \in \mathcal{H}$. Define the diffeomorphism $\lambda_3(h)$ by the rule: if $h(E_{ri}) = E_{rj}$, then

$$\lambda_3(h)|_{E_{ri}} = \tilde{h}_{rj} \circ \tilde{h}_{ri}^{-1} : E_{ri} \xrightarrow{\tilde{h}_{ri}^{-1}} E_{r0} \xrightarrow{\tilde{h}_{rj}} E_{rj}.$$

It follows from Lemma 5.5 that $\lambda_3(h)$ coincides with $\lambda_2(h)$ on Y .

Now we will check that the correspondence $h \mapsto \lambda_3(h)$ is a homomorphism. Let h_1 and h_2 be homeomorphisms from \mathcal{H} such that $h_1(E_{ri}) = E_{rj}$ and $h_2(E_{rj}) = E_{rk}$. By definition $\lambda_3(h_1)|_{E_{ri}} = \tilde{h}_{rj} \circ \tilde{h}_{ri}^{-1}$, and $\lambda_3(h_2)|_{E_{rj}} = \tilde{h}_{rk} \circ \tilde{h}_{rj}^{-1}$. Then $\lambda_3(h_2)|_{E_{rj}} \circ \lambda_3(h_1)|_{E_{ri}} = \tilde{h}_{rk} \circ \tilde{h}_{rj}^{-1} \circ \tilde{h}_{rj} \circ \tilde{h}_{ri}^{-1} = \tilde{h}_{rk} \circ \tilde{h}_{ri}^{-1} = \lambda_3(h_2 \circ h_1)|_{E_{ri}}$. Hence, the map λ_3 is a homomorphism.

Let $g \in H$, and $h \in \mathcal{H}$ be such that $\phi(h) = g$. By condition (C), $\lambda_3(h) = \text{id}$ iff $h(E_{ri}) = E_{ri}$ for some $r = 1, 2, \dots, y$, and $i = 0, 1, \dots, k(r)$. So the map $\chi_3 : H \rightarrow \text{Diff}(\mathbf{E})$, defined by $\chi_3(g) = \lambda_3(h)$, is a monomorphism.

Let $\sigma' : H \times \mathbf{E} \rightarrow \mathbf{E}$ be the map given by the formula

$$\sigma'(g, x) = \chi_3(g)(x), \quad x \in \mathbf{E}.$$

Since χ_3 is a homomorphism, it follows that σ' is an H -action on \mathbf{E} .

Hence, we define an H -action $\Sigma : H \times M \rightarrow M$ on M by the rule:

$$\Sigma = \begin{cases} \sigma', & \text{on } H \times \mathbf{E}, \\ \sigma, & \text{on } H \times \mathcal{E}_{V,a}. \end{cases}$$

Theorem 2.2 is proved. \square

Proof of Lemma 5.4. Due to [9, Lemma 4.12.], for the diffeomorphism $h'_{ri} = \chi_2^{-1}(g)|_{Y_{ri}} \circ h_{ri}|_{Y_r} : Y_r \rightarrow Y_r$ of the cylinder there exists a smooth function α_{ri} such that $h'_{ri} = \mathbf{F}_{\alpha_{ri}}$ whenever for each trajectory ω of \mathbf{F} we have that $h'_{ri}(\omega) = \omega$ and h'_{ri} preserves orientation of ω .

Let ω be a trajectory of \mathbf{F} . It follows from condition (C) that $\omega = f^{-1}(t) \cap Y_r$ for some $t \in \mathbb{R}$. The sets $f^{-1}(t)$ and Y_r are h'_{ri} -invariant, so the set $f^{-1}(t) \cap Y_r$ is also h'_{ri} -invariant. Hence, $h'_{ri}(\omega) = \omega$ for all trajectories of \mathbf{F} . Moreover, by Lemma 4.1, h'_{ri} preserves orientations of each orbit ω of \mathbf{F} . Thus $h'_{ri} = \mathbf{F}_{\alpha_{ri}}$, and due to (5.3) we put $\xi_{ri} = -\alpha_{ri} \circ \mathbf{F}_{\alpha_{ri}}^{-1}$. Lemma is proved.

Proof of Lemma 5.5. By the result of Seeley [14], the function ξ_{ri} extends to some smooth function β'_{ri} on E_{ri} . It is easy to construct a function $\delta_r \in C^\infty(E_{r0}, [0, 1])$ which satisfies the following conditions:

- (1) $\delta_r = 1$ on Y_r ,
- (2) $\delta_r = 0$ on some neighborhood of $Z_r \cap (f^{-1}(-a_1) \cup f^{-1}(a_1))$.
- (3) $F\delta_r = 0$, i.e., δ_r is constant along orbits of F ,
- (4) the function $\beta_{ri} = \delta_r \beta'_{ri}$, $i = 1, 2, \dots, n$ satisfies the inequality $F\beta_{ri}|_{Z_r} > -1$.

Indeed, since $\beta'_{ri} = \xi_{ri}$ on Y_{ri} and $\mathbf{F}_{\alpha_{ri}}$ is a diffeomorphism, it follows that $F\beta'_{ri} > -1$ on Y_{ri} . Then there exists $b \in (a, a_1)$ such that $F\beta'_{ri} > -1$ on $A_r = E_{r0} \cap f^{-1}([-b, -a/2] \cup [a/2, b]) \cap \mathcal{E}_{V,a}$. Let $\delta_r : E_{r0} \rightarrow [0, 1]$ be a smooth function such that $\delta_r = 1$ on Y_r , $\delta_r = 0$ on $E_{r0} \setminus A_r$, and $F\delta_r = 0$. Then $\delta F\beta'_{ri} > -1$ on E_{ri} . Now the required diffeomorphism $w_{ri} : E_{r0} \rightarrow E_{r0}$ can be defined by the formula

$$w_{ri}(x) = \begin{cases} \mathbf{F}_{\beta_{ri}}, & x \in Z_r, \\ x, & x \in E_{r0} \setminus Z_r. \end{cases}$$

Lemma is proved.

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REFERENCES

1. A. V. Bolsinov and A. T. Fomenko, *Some actual unsolved problems in topology of integrable Hamiltonian systems*, Topological classification in theory of Hamiltonian systems, Factorial, Moscow, 1999, pp. 5–23. (Russian)
2. Yu. Brailov, *Algebraic properties of symmetries of atoms*, Topological classification in theory of Hamiltonian systems, Factorial, Moscow, 1999, pp. 24–40. (Russian)
3. Yu. A. Brailov and E. A. Kudryavtseva, *Stable topological nonconjugacy of Hamiltonian systems on two-dimensional surfaces*, Moscow Univ. Math. Bull. **54** (1999), no. 2, 20–27.
4. E. N. Dancer, *Degenerate critical points, homotopy indices and Morse inequalities*, J. Reine Angew. Math. **350** (1984), 1–22.
5. E. A. Kudryavtseva, *Realization of smooth functions on surfaces as height functions*, Mat. Sb. **190** (1999), no. 3, 29–88.
6. E. A. Kudryavtseva and A. T. Fomenko, *Symmetry groups of nice Morse functions on surfaces*, Dokl. Akad. Nauk **446** (2012), no. 6, 615–617.
7. E. A. Kudryavtseva and A. T. Fomenko, *Each finite group is a symmetry group of some map (an “Atom”-bifurcation)*, Moscow Univ. Math. Bull. **68** (2013), no. 3, 148–155.
8. Sergey Maksymenko, *Smooth shifts along trajectories of flows*, Topology Appl. **130** (2003), no. 2, 183–204.
9. Sergiy Maksymenko, *Homotopy types of stabilizers and orbits of Morse functions on surfaces*, Ann. Global Anal. Geom. **29** (2006), no. 3, 241–285.
10. Sergiy Maksymenko, *Connected components of partition preserving diffeomorphisms*, Methods Funct. Anal. Topology **15** (2009), no. 3, 264–279.
11. Sergiy Maksymenko, *Deformations of functions on surfaces by isotopic to the identity diffeomorphisms*, arXiv:math/1311.3347, 2013.
12. A. A. Oshemkov, *Morse functions on two-dimensional surfaces. Coding of singularities*, Trudy Mat. Inst. Steklov. **205** (1994), no. Novye Rezult. v Teor. Topol. Klassif. Integr. Sistem, 131–140.
13. A. O. Prishlyak, *Topological equivalence of smooth functions with isolated critical points on a closed surface*, Topology Appl. **119** (2002), no. 3, 257–267.
14. R. T. Seeley, *Extension of C^∞ functions defined in a half space*, Proc. Amer. Math. Soc. **15** (1964), 625–626.

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